An invitation to higher dimensional partitions

Suresh Govindarajan
suresh@iitm.ac.in

Department of Physics
Indian Institute of Technology Madras

ICNG 2019 on June 28 at U. of Mysuru

Happy birthday Prof. Adiga

Plan

Introduction

The partition function

- A partition of an integer $n > 0$ is to express it as a sum of positive integers. For instance, $2 + 1 + 1$ is a partition of the integer 4.
- As $1 + 2 + 1$ and $2 + 1 + 1$ are the same partition, one chooses to write it as a *weakly decreasing* sequence $(2, 1, 1)$ to get a unique representative for each partition.
- Let $p(n)$ denote the number of partitions of n. For $n = 4$, one has

$$4 \quad 3\,1 \quad 2\,2 \quad 2\,1\,1 \quad 1\,1\,1\,1 \implies \boxed{p(4) = 5}\ .$$

  $p(n)$ is called the partition function.
- Define the generating function, $P(q) := 1 + \sum_{n=1}^{\infty} p(n)\ q^n$. Euler showed that

$$P(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-1}\ .$$

Higher-dimensional partitions

- A *d-dimensional partition* of n is defined to be a map from $\mathbb{Z}^d_{>0}$ to $\mathbb{Z}_{\geq 0}$ such that it is weakly decreasing along all directions and the sum of all its entries add to n.
- Let us denote the partition by the hypermatrix $(a_{i_1,i_2,\ldots,i_d})$.
- The weakly decreasing condition along the r-th direction implies that

$$a_{i_1,i_2,\ldots,i_r+1,\ldots,i_d} \leq a_{i_1,i_2,\ldots,i_r,\ldots i_d} \quad \forall\ (i_1, i_2, \ldots, i_d) \in \mathbb{Z}^d_{>0}\ .$$

- Let us denote the d-dimensional partition of n by $p_d(n)$. Thus $p_1(n)$ is the partition function.
- Two-dimensional partitions are also called plane partitions.
- Three-dimensional partitions are also called solid partitions.

Plane partitions

- Plane partitions can thus be written out as a two-dimensional array of numbers, (a_{ij}).
- For instance, the plane partitions of 4 are

$$
4 \quad 3\;1 \quad \begin{matrix}3\\1\end{matrix} \quad 2\;2 \quad \begin{matrix}2\\2\end{matrix} \quad 2\;1\;1 \quad \begin{matrix}2&1\\1&\end{matrix} \quad \begin{matrix}2\\1\\1\end{matrix}
$$

$$
1\;1\;1\;1 \quad \begin{matrix}1&1&1\\1&&\end{matrix} \quad \begin{matrix}1&1\\1&1\end{matrix} \quad \begin{matrix}1&1\\1&\\1&\end{matrix} \quad \begin{matrix}1\\1\\1\\1\end{matrix} \qquad \Longrightarrow \quad \boxed{p_2(4) = 13}
$$

- [MacMahon] Let $P_2(q) := 1 + \sum_{n=1} p_2(n)\, q^n$. Then

$$
P_2(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-m} \ .
$$

- Took MacMahon about 20 years to prove his conjecture! There exists no elementary proof to date.

Generating Functions for $d > 2$

- Define the generating function for a d-dimensional partition as

$$P_d(q) = 1 + \sum_{n=1} p_d(n)\ q^n \ .$$

- MacMahon also conjectured formulae for the generating function for solid and other higher dimensional partitions. Let

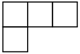
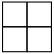
$$M_d(q) := \prod_{m=1}^{\infty} (1 - q^m)^{-\binom{n+d-2}{d-1}} = 1 + \sum_{n=1}^{\infty} m_d(n)\ q^n \ .$$

  It was shown in 1967 by Atkin et. al. that this fails i.e., $m_d(n) \neq p_d(n)$ for $d > 2$ and $n \geq 6$.

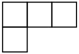
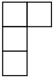
- It appears that there is no simple formula for the generating function.

From partitions to Young diagrams

- Given a partition, $(a_1, a_2, a_3, \ldots)$, draw a Young diagram with a_k-boxes in the k-th row. For instance,

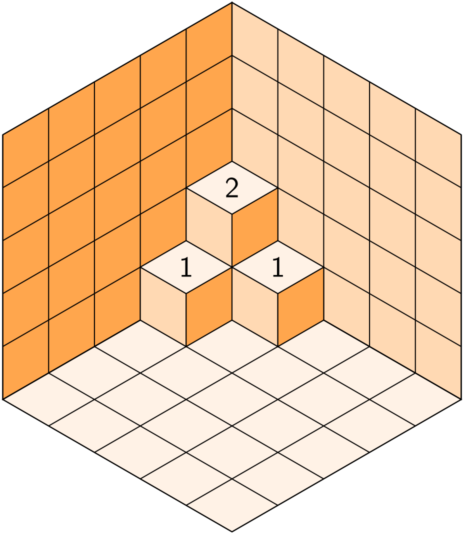
3 1 $\longleftrightarrow$  ; 2 2 $\longleftrightarrow$ 

- It is clear that this map is a bijection. Given a Young diagram, one can obtain the corresponding partition by counting the number of boxes in each row.
- There is an involution that acts on Young diagrams called conjugation. It corresponds to the xy-flip.

(3 1) $=$  $\longleftrightarrow$  $(= 2\ 1\ 1)$

- Viewed as acting on partitions, we see that it maps a partition with r-parts to one with largest part r. This is Ferrer's bijective proof showing that $p(n|r \text{ parts}) = p(n|\text{l.p.} = r)$.

Plane Partitions as 3d Young Diagrams



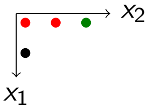
Ferrers diagrams

- It is clear how to generalise Young diagrams to higher dimensions replacing cubes in 3D to hypercubes in 4D and so on. However, visualisation is not possible.
- We replace the squares/cubes/... with integral points in $\mathbb{R}_+^{d+1}$ – call the points nodes.
- An unrestricted d-dimensional partition of n is a collection of n points (nodes) in $\mathbb{Z}_{\geq 0}^{d+1}$ satisfying the following property: if the collection contains a node $\mathbf{a} = (a_1, a_2, \ldots, a_{d+1})^T$, then all nodes $\mathbf{x} = (x_1, x_2, \ldots, x_{d+1})^T$ with $0 \leq x_i \leq a_i$ $\forall$ $i = 1, \ldots, d+1$ also belong to it.
- For instance, the following is a one-dimensional partition of 4

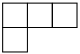
$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \text{ or } \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{smallmatrix} \right) \text{ in compressed form },$$

Ferrers diagrams

- It is clear how to generalise Young diagrams to higher dimensions replacing cubes in 3D to hypercubes in 4D and so on. However, visualisation is not possible.
- We replace the squares/cubes/... with integral points in $\mathbb{R}_+^{d+1}$ – call the points nodes.
- An unrestricted d-dimensional partition of n is a collection of n points (nodes) in $\mathbb{Z}_{\geq 0}^{d+1}$ satisfying the following property: if the collection contains a node $\mathbf{a} = (a_1, a_2, \ldots, a_{d+1})^T$, then all nodes $\mathbf{x} = (x_1, x_2, \ldots, x_{d+1})^T$ with $0 \leq x_i \leq a_i$ $\forall$ $i = 1, \ldots, d+1$ also belong to it.
- The same collection of nodes can be viewed as a Ferrers diagram or a Young diagram.



or



History of Higher Dimensional Partitions

After MacMahon, the first serious computation of higher dimensional partitions, due to Atkin, Bratley, MacDonald and McKay, appeared in 1967. Here is a report by Birch on this paper in his memoir on Atkin:

I cannot resist mention [1967d], on m-dimensional partitions. At the time the authors complained that no one seemed to know anything about them except in the first two cases (ordinary partitions corresponding to the case m=2, and the case m=3 more often known as plane partitions); and very little seems to have been discovered since; there is a note on the subject in [1971d]. In the words of the third author, the paper landed like a lead balloon; but they look genuinely interesting.

Stanley in his 1971 doctoral thesis writes:

The case $r = 2$ has a well-developed theory – here 2-dimensional partitions are known as plane partitions. See 21 and the survey article by Stanley[34] for results on plane partitions. For $r \geq 3$, almost nothing is known and Proposition 11.1 casts only a faint glimmer of light on a vast darkness.

Refined counting of higher-dimensional partitions

SG, *Notes on higher-dimensional partitions*, J. Comb. Theory Ser. A **120**, 600-622, (2013)

Conjugation in higher-dimensional partitions

- The analog of conjugation is now the permutation group S_{d+1} – it permutes the $(d+1)$ axes in the Ferrers's diagram.
- An important simplification occurs if we treat partitions in all dimensions on the same footing.
- Consider the FD for the partition of 2 in 1/2/3 dimensions.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

  The nodes lie in the one-dimensional hyperplane $x_i = 0$ for $i = 2, 3, \ldots$. The only non-zero coordinate is x_1.
- Def: The intrinsic dimension of an FD is defined to be the minimal dimension of hyperplane that contains all its nodes.
- Observe that any partition of 2 in any dimension has id $= 1$.

Proposition (Atkin, Bratley, McKay and MacDonald, 1967)

Define $a_{n.r}$ *via the transform*

$$p_d(n) = \sum_{r=0}^{n-1} \binom{d+1}{r} a_{n,r} \ ,$$

where $a_{n,r}$ *enumerates the number of* $(r-1)$ *dimensional partitions of* n *with id*$=r$*.*

- $a_{n0} = \delta_{n,1}$ – this follows since there is precisely one FD with $id = 0$: □. It has $n = 1$.
- $a_{r+1,r} = 1$ for all $n \geq 1$ – again there is only one FD of size $(r+1)$ and id r.
- $a_{n,r} = 0$ when $r \leq n$. It is impossible to construct a FD of id r with fewer than $r+1$ nodes.

The triangle A

This implies that the $(n-1)$ non-zero numbers in $a_{n,r}$, i.e., a row of the triangular matrix A determines partitions of n in all dimensions.

$$A = (a_{n,r}) = \begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 5 & 6 & 1 \\ 0 & 1 & 9 & 18 & 10 & 1 \\ 0 & 1 & 13 & 44 & 49 & 15 & 1 \\ 0 & 1 & 20 & 97 & 172 & 110 & 21 & 1 \\ 0 & 1 & 28 & 195 & 512 & 550 & 216 & 28 & 1 \\ 0 & 1 & 40 & 377 & 1370 & 2195 & 1486 & 385 & 36 & 1 \\ 0 & 1 & 54 & 694 & 3396 & 7603 & 7886 & 3514 & 638 & 45 & 1 \end{pmatrix}$$

Bratley and McKay created an algorithm to enumerate partitions in arbitrary dimensions. In the paper with Atkin and MacDonald, they obtained the first few rows of the A-matrix.

Proposition (SG, 2013)

Define $F = (f_{n,x})$ through the transform of the matrix A.

$$a_{m+r+1,r} = \sum_{x=0}^{r} \sum_{p=x}^{m} \binom{r}{x} \binom{\binom{r-x}{2}}{m-p} f_{p+x+1,\ x} \,.$$

- The main result is that the n-th row of F has only $[(n-1)/2]$ non-zero numbers.
- We illustrate the gain by explicitly displaying the first eleven rows of the A and F-matrices.

$$A = \begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 5 & 6 & 1 \\ 0 & 1 & 9 & 18 & 10 & 1 \\ 0 & 1 & 13 & 44 & 49 & 15 & 1 \\ 0 & 1 & 20 & 97 & 172 & 110 & 21 & 1 \\ 0 & 1 & 28 & 195 & 512 & 550 & 216 & 28 & 1 \\ 0 & 1 & 40 & 377 & 1370 & 2195 & 1486 & 385 & 36 & 1 \\ 0 & 1 & 54 & 694 & 3396 & 7603 & 7886 & 3514 & 638 & 45 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 \\ 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 7 \\ 0 & 1 & 11 & 16 \\ 0 & 1 & 18 & 58 \\ 0 & 1 & 26 & 135 & 125 \\ 0 & 1 & 38 & 293 & 618 \\ 0 & 1 & 52 & 574 & 1927 & 1296 \end{pmatrix}.$$

- We have determined the rows of the F-matrix for $n \leq 26$.

Asymptotics of solid partitions

N. Destainville and SG, *Estimating the asymptotics of solid partitions*, J. Statistical Phys. **158**, 950-967, (2015)

Proposition (Bhatia, Prasad and Arora (1997))

For two non-negative real (d-dependent) constants, $c_L < c_U$, for large n

$$c_L < n^{-d/(d+1)} \log p_d(n) < c_U \ .$$

- Since the generating functions for ordinary and plane partitions are known, one can show that

$$p_1(n) \sim \frac{1}{4\sqrt{3}\ n} \exp\left(\pi\sqrt{\tfrac{2}{3}}\ n^{1/2}\right)$$

$$p_2(n) \sim \frac{\zeta(3)^{7/36}}{\sqrt{12\pi}} \left(\frac{n}{2}\right)^{-25/36} \exp\left(3\ \zeta(3)^{1/3}\left(\frac{n}{2}\right)^{2/3} + \zeta'(-1)\right)$$

Proposition (Bhatia, Prasad and Arora (1997))

For two non-negative real (d-dependent) constants, $c_L < c_U$, for large n

$$c_L < n^{-d/(d+1)} \log p_d(n) < c_U \ .$$

- Since the generating functions for ordinary and plane partitions are known, one can show that

$$n^{-1/2} \log p_1(n) \sim 2.5651 - n^{-1/2} (\log n + \text{constant})$$
$$n^{-2/3} \log p_2(n) \sim 2.00945 - n^{-2/3} (0.69444 \log n + 1.46310) \ .$$

Proposition (Bhatia, Prasad and Arora (1997))

For two non-negative real (d-dependent) constants, $c_L < c_U$, for large n

$$c_L < n^{-d/(d+1)} \log p_d(n) < c_U \ .$$

- Since the generating functions for ordinary and plane partitions are known, one can show that

$$n^{-1/2} \log p_1(n) \sim 2.5651 - n^{-1/2} (\log n + \text{constant})$$
$$n^{-2/3} \log p_2(n) \sim 2.00945 - n^{-2/3} (0.69444 \log n + 1.46310) \ .$$

- We conjecture (with $\ell := n^{1/(d+1)}$)

$$\log p_d(n) \sim \mathcal{C}_d \ \ell^d + \beta_d^{(1)} \ \ell^{(d-1)} + \cdots + \beta_d^{(d)} + \gamma_d \ \log \ell + O(\ell^{-1})$$

- For $d = 2$, $\beta_2^{(1)} = 0$ but one cannot assume that it will be true for $d > 2$.

The asymptotics of solid partitions

- We used a Markov Chain Monte Carlo simulation to estimate the asymptotics of solid partitions. The simulation estimates the ratio $p_3(n)/p_3(n-1)$.
- We look for an asymptotic formula for $p_3(n)$ of the form

$$\log p_3(n) \sim \alpha_3\ n^{3/4} + \beta_3\ n^{2/4} + \gamma_3\ n^{1/4} + \delta_3 \log n + \epsilon_3\ .$$

- This implies

$$\log p_3(n) - \log p_3(n-1) \sim \frac{3\alpha_3}{4}\ n^{-1/4} + \frac{\beta_3}{2}\ n^{-1/2} + \frac{\gamma_3}{4}\ n^{-3/4} + \delta_3\ n^{-1}\ .$$

  Note that the parameter ϵ_3 drops out and hence has to be determined independently.
- The simulations over the range $n \in [50, 10100]$ were used to fix the parameters.

Results

- Fit data in the range $n \in [50, 10100]$ to the four-parameter formula. This determines the values of $(\alpha_3, \beta_3, \gamma_3, \delta_3)$ that we will use along. We find

$$(\alpha_3, \beta_3, \gamma_3, \delta_3) = (1.82228 \pm 0.00004, 0.06136 \pm 0.0008, \\ 0.999 \pm 0.008, -0.828 \pm 0.003) \ .$$

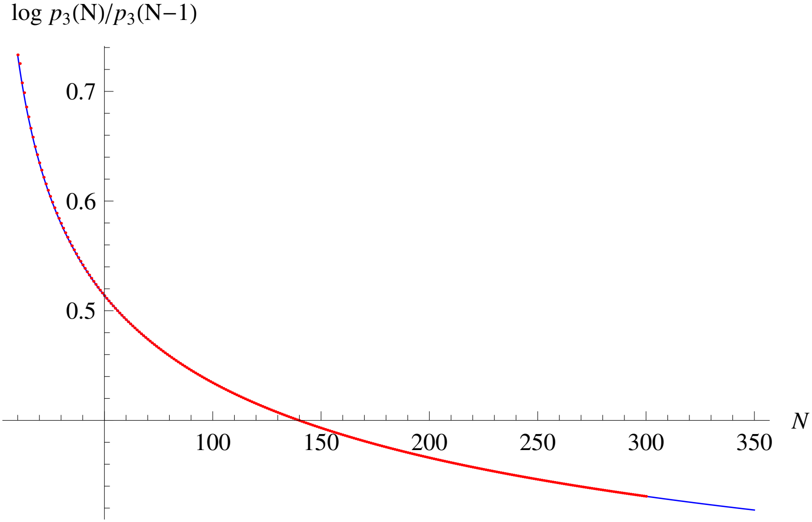
- Next we add the term $\left[-\frac{1}{4} f \;\; n^{-5/4} \right]$ and carry out a five-parameter fit to see how the four parameters change.

$$(\alpha_3, \beta_3, \gamma_3, \delta_3, f) = (1.8215 \pm 0.0001, 0.088 \pm 0.004, \\ 0.60 \pm 0.06, -0.51 \pm 0.04, 1.44 \pm 0.19) \ .$$

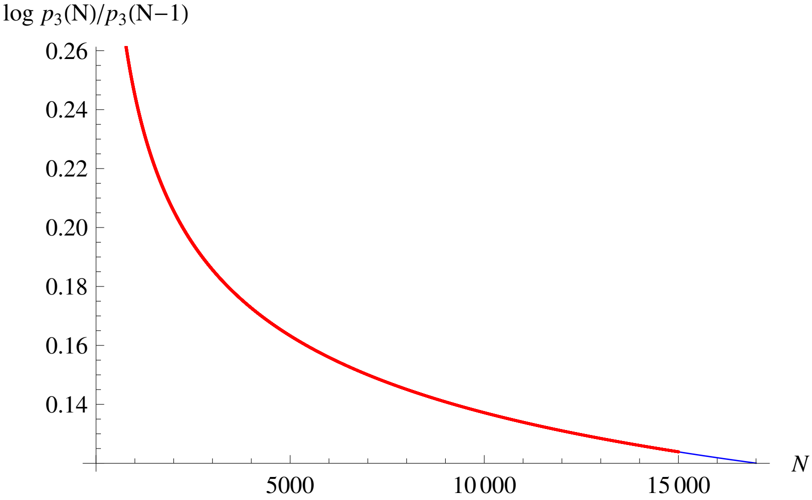
We conclude

$$(\alpha_3, \beta_3, \gamma_3, \delta_3) = (1.822 \pm 0.001, 0.06 \pm 0.03, \\ 1.0 \pm 0.4, -0.8 \pm 0.3) \ .$$

How good are our fits?



How good are our fits?



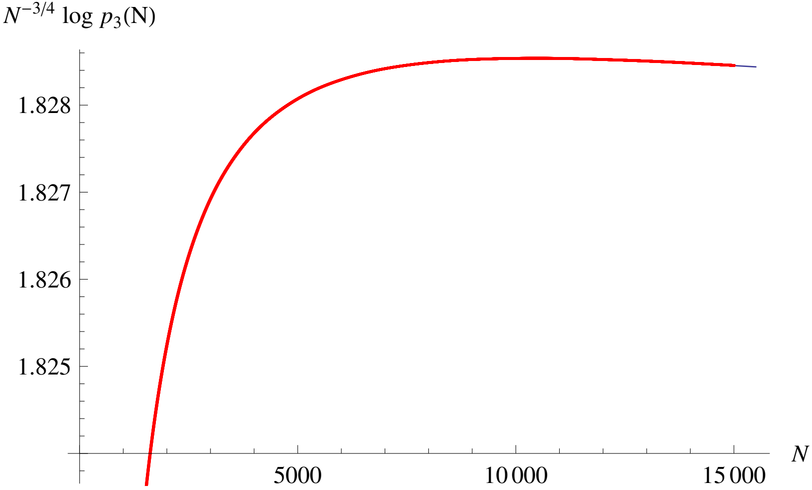
Determining ϵ_3

- We can estimate $p_3(n)$ from the Monte Carlo by starting with, say $p_3(50)$, which is known exactly. Let us call this estimate $mc_3(n)$.
- To determine ϵ_3, we work with $mc_3(n)$ and the values of $(\alpha_3, \beta_3, \gamma_3, \delta_3)$ that we determined.
- We carry out a one-parameter fit for $n \in [50, 100]$ (our best quality data) to obtain ϵ_3. We obtain $\epsilon_3 = -2.24385$.
- The number is fairly robust if we add more data, say up to 300.
- We then add a term $f\ n^{-1}$ to the asymptotic formula for $p_3(n)$ and carry out a two-parameter fit determine ϵ_3. We obtain $\epsilon_3 = -3.12961$. We thus obtain the estimate

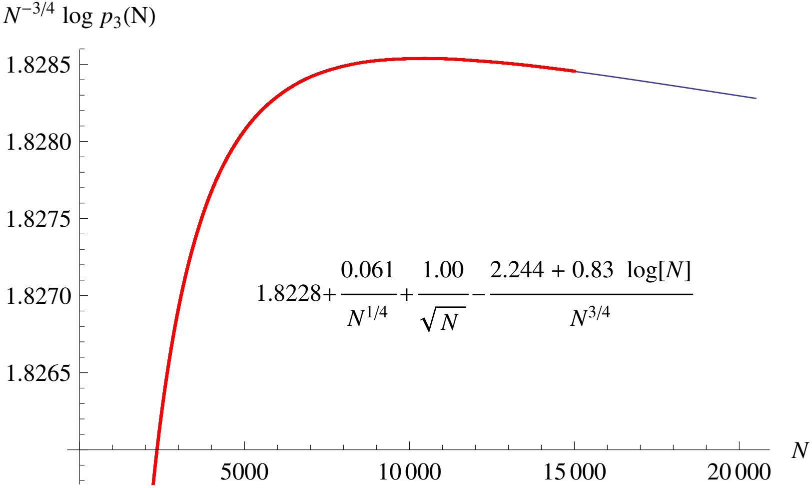
$$\epsilon_3 = -2.2 \pm 0.9 \ .$$

where we used the change in ϵ_3 to fix the error estimate.

How good are our fits?



How good are our fits?

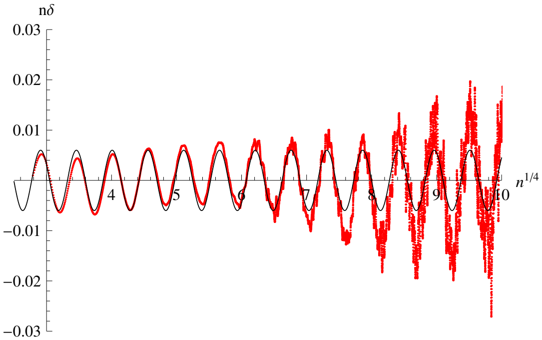


The appearance of oscillations

Define the residual

$$\delta := \log\left[\frac{mc_3(n)}{mc_3(n-1)}\right] - \left(\alpha_3\left[n^{3/4}\right]_3 + \beta_3\ \left[n^{2/4}\right]_3 + \gamma_3\ \left[n^{1/4}\right]_3 + \delta_3\left[\log n\right]_3\right)$$

with $(\alpha_3, \beta_3, \gamma_3, \delta_3)$ as given by our fit.



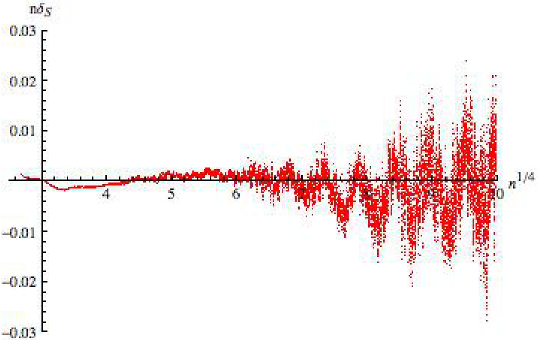
The black curve is $0.006\cos[2\pi(1.817n^{1/4} - 0.29)]$.

The appearance of oscillations

Define the residual

$$\delta := \log\left[\frac{mc_3(n)}{mc_3(n-1)}\right] - \left(\alpha_3\left[n^{3/4}\right]_3 + \beta_3\ \left[n^{2/4}\right]_3 + \gamma_3\ \left[n^{1/4}\right]_3 + \delta_3\left[\log n\right]_3\right)$$

with $(\alpha_3, \beta_3, \gamma_3, \delta_3)$ as given by our fit.



$n\delta_S$ is $n\delta$ after subtracting the black curve.

Conclusion on Solid Partitions Asymptotics

- Similar oscillations were not observed in the Monte Carlo simulations for plane partitions.
- We need to use the following asymptotic formula for solid partitions:

$$\log p_3(n) \sim \alpha_3\ \xi^{3/4} + \beta_3\ \xi^{2/4} + \gamma_3\ \xi^{1/4} + \delta_3 \log \xi + \epsilon_3 + \xi^{-1/4}\Big(g \sin[2\pi\nu\xi^{1/4} + \varphi]\Big) + \cdots\ ,$$

  where $\xi = n + \zeta$.
- The new parameters are indicated in red and are sub-leading to the constant term, ϵ_3. We need better data to estimate them properly though our graphical approach has given them some value.
- It is an open problem to explain the appearance of oscillations.

An HRR type formula for the plane partition function

A superasymptotic formula for the number of plane partitions

SG and Naveen S. Prabhakar,

arXiv:1311.7227 [math.NT].

From generating functions to partition functions

- The generating functions, $P(q)$ and $P_2(q)$, can be used to get 'formulae' for the partition function, $p(n)$, and the plane partition function, $p_2(n)$
- For instance, one can prove the following recurrence relations:
$$p(n) = \frac{1}{n} \sum_{m=1}^{n} \sigma_1(m) p(n-m) \ ,$$
where $\sigma_x(m) = \sum_{d|m} d^x$ is the divisor function.
- A complex analytic way is to use the Cauchy residue theorem to write (C is a counter-clockwise contour around $q = 0$)
$$p(n) = \frac{1}{2\pi i} \oint_C \frac{dq}{q^{n+1}} P(q)$$
- Hardy and Ramanujan invented the circle method to evaluate the above integral to obtain their formula for $p(n)$.

From generating functions to partition functions

- The generating functions, $P(q)$ and $P_2(q)$, can be used to get 'formulae' for the partition function, $p(n)$, and the plane partition function, $p_2(n)$
- For instance, one can prove the following recurrence relations:

$$p_2(n) = \frac{1}{n} \sum_{m=1}^{n} \sigma_2(m) p_2(n-m) ,$$

where $\sigma_x(m) = \sum_{d|m} d^x$ is the divisor function.
- A complex analytic way is to use the Cauchy residue theorem to write (C is a counter-clockwise contour around $q = 0$)

$$p_2(n) = \frac{1}{2\pi i} \oint_C \frac{dq}{q^{n+1}} P_2(q)$$

- Similarly, we use the above integral to obtain a formula for $pl(n)$.

The circle method – an overview

- One first observes that $P(q)$ is analytic inside the unit circle in the q-plane.
- All singularities lie on the unit circle $|q|$ and appear at roots of unity ie., for $q = \exp(2\pi i h/k)$ for all integers $k \geq 1$ and $0 < h < k$ with $(h, k) = 1$ ('the (h, k)-th root of unity').
- Expanding $P(q)$ in the neighbourhood of the (h, k)-th root of unity and carry out the integral in that neighbourhood.
- Carry out the sum over all (h, k) to formally obtain a formula:

$$p(n) = \lim_{N\to\infty} \sum_{k=1}^{N} \sum_{\substack{h=1 \\ (h,k)=1}}^{k-1} \text{the } (h,k)\text{-th root of unity's contribution} \ .$$

- Remark: The singularities of $PL(q)$ also appear at all (h, k)-th roots of unity. Hence similar methods can be used.

The circle method – details[1]

- To focus on the neighbourhood of a particular root of unity, define $q = \exp(2\pi i \frac{h}{k} - z)$ and let $|q| = \exp(-\rho(N)) < 1$ for some function $\rho(N)$ that tends to zero as $N \to \infty$. For $p(n)$, we choose $\rho(N) = \frac{2\pi}{N^2}$.
- It is very useful that $P(q)$ is related to the Dedekind eta function. One has $\eta(\tau) = q^{1/24} P(q)^{-1}$ with $q = \exp(2\pi i \tau)$.
- Using this one obtains a formula for $P(q)$ in the neighbourhood of the (h, k)-th root of unity as $z \to 0$ and $\mathrm{Re}(z) > 0$.

$$\widehat{P}\left(\exp(\tfrac{2\pi i h}{k} - \tfrac{2\pi z}{k})\right) = e^{is(h,k)} z^{1/2} \exp\left[\frac{\pi(-z + z^{-1})}{12k}\right] \times \left(1 + O(e^{-2\pi/(kz)})\right)$$

where $s(h, k) = \sum_{m=1}^{k-1} ((\frac{m}{k}))((\frac{mh}{k}))$ is the Dedekind sum, This representation of the phase $e^{is(h,k)}$, which is a 24k-th root of unity, is due to Rademacher.

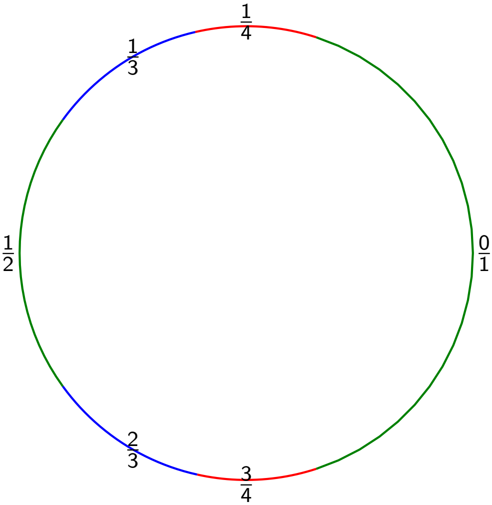
[1] We follow G.E. Andrews, 'Theory of Partitions', Cambridge U. Press

The circle method – dissecting the circle

- Let $\mathcal{F}_N$ denote the Farey sequence of order N for some positive integer. For example: $\mathcal{F}_4 = (\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1})$
- The circle method then leads to a sum over all fractions $h/k \in \mathcal{F}_N$ with each term picking the contribution of the pole at $q = \exp(2\pi i h/k)$.
- After a change of variable $q = \exp(2\pi i h/k)\, e^{-z}$ in the neighbourhood of $q = \exp(2\pi i h/k)$, one has $\frac{dq}{q} = -dz$.
- The neighbourhood of the (h, k)-th root of unity is given by $z \in [z'_{h,k}, z''_{h,k}]$ where $z'_{h,k} = \rho(N) + 2\pi i \zeta'_{h,k}$ and $z''_{h,k} = \rho(N) - 2\pi i \zeta''_{h,k}$ with
$$\zeta'_{0,1} = \frac{1}{N+1}\ ,$$
$$\zeta'_{h,k} = \frac{h}{k} - \frac{h_0+h}{k_0+k} \quad \text{for} \quad h > 0\ ,$$
$$\zeta''_{h,k} = \frac{h+h_1}{k+k_1} - \frac{h}{k}\ ,$$
where h_0/k_0, h/k and h_1/k_1 are successive terms in $\mathcal{F}_N$.
- The circle of radius $|q| = e^{-\rho(N)} < 1$ is thus split into segments for each $(h, k) \in \mathcal{F}_N$. ($\rho(N) = 2\pi/N^2$ for $p(n)$.)

Dissection of the circle for $N = 4$

$$\left(\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right)$$



The circle method – putting things together

- The circle method provides the following expression for $p(n)$:

$$p(n) = \lim_{N\to\infty} \sum_{(h,k)\in\mathcal{F}_N} \int_{z'_{h,k}}^{z''_{h,k}} \frac{\mathrm{d}z}{-2\pi i}\ e^{nz-2\pi i n\frac{h}{k}}\ \widehat{P}\left(e^{2\pi i\frac{h}{k}} e^{-z}\right)\ ,$$

$$=: \lim_{N\to\infty} \sum_{(h,k)\in\mathcal{F}_N} \psi_{h,k}(n) =: \sum_{k=1}^{\infty} \phi_k(n)\ .$$

where (as $z \to 0$ and $\mathrm{Re}(z) > 0$)

$$\widehat{P}\left(e^{\frac{2\pi i h}{k}-z}\right) = e^{is(h,k)}\left(\tfrac{zk}{2\pi}\right)^{1/2} \exp\left[-\tfrac{z}{24} + \tfrac{\pi^2}{6zk^2}\right]\left(1 + O(e^{-\frac{4\pi^2}{zk}})\right)$$

- We use the hat to remind one that the right hand side is valid as $z \to 0$ and $\mathrm{Re}(z) > 0$.
- All that remains to be done is to carry out the integration.

The Hardy-Ramanujan formula

- Heuristically, one observes that the dominant contribution to $p(n)$ comes from the $k = 1$ term i.e., the pole at $q = 1$.
- Hardy and Ramanujan carried out the $k = 1$ integral by the saddle-point method and obtained

$$\phi_1(n) = \psi_{0,1}(n) \sim \frac{1}{4n\sqrt{3}} \, \exp\left(\sqrt{\frac{2\pi^2 n}{3}}\right) \; .$$

- Comparing with actual value of $p(n)$ for low values such as $n = 10, 20, 50$, they observed that it always an over-estimate.
- They looked for a modification which matched the above formula at large n and came up with an inspired modification.

$$\phi_1(n) = \psi_{0,1}(n) \sim \frac{1}{2\pi\sqrt{2}} \left[\frac{d}{dx} \frac{\exp \pi(\frac{2}{3}(x - \frac{1}{24}))^{1/2}}{(x - \frac{1}{24})^{1/2}}\right]_{x=n} \; .$$

The Hardy-Ramanujan formula

- This worked rather well!

n	$p(n)$	$\phi_1(n)$	$p(n) - \phi_1(n)$
61	1121505	1121539.172	-34.172
62	1300156	1300121.578	34.4217
63	1505499	1505535.605	-36.6048

- The oscillating pattern in the error is precisely what one expects from the phase associated with the pole at $q = -1$. So they came up with a formula that includes the contribution of all poles.

$$p(n) \sim \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\ k^{1/2} \left[\frac{d}{dx} \frac{\exp \frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{1/2}}{\left(x - \frac{1}{24}\right)^{1/2}} \right]_{x=n}$$

where

$$A_k(n) = \sum_{\substack{h=1 \\ (h,k)=1}}^{k-1} e^{\pi i s(h,k)}\ e^{-2\pi i n h/k}$$

The Hardy-Ramanujan formula

- Hardy writes "At this point we may have stopped had it not been for Major MacMahon's love of calculation. MacMahon was a practised and enthusiastic computer, and made us a table of up to $n = 200$. In particular, he found $p(200) = 3972999029388$. We naturally took this value as a test for our asymptotic formula. We expected a good result, with an error of perhaps one or two figures, but we had never dared for such a result as we found."

+ 3,972,998,993,185.896
+ 36,282.978
− 87.555
+ 5.147
+ 1.424
+ 0.071
+ 0.000
+ 0.043
3,972,999,029,388.004

- The formula is however not a convergent series.

The Hardy-Ramanujan-Rademacher Formula for $p(n)$

- The major issue with the HR formula is that the sum over k is not convergent and the series is asymptotic.
- Rademacher was preparing to lecture on the HR formula for $p(n)$ in his course and started working through the proof. He discovered that a simple modification of the HR formula leads to a convergent sum! This proved the existence of an exact formula for $p(n)$ that Ramanujan always expected.
- In simple words, the change is to replace $\frac{1}{2}\exp(x)$ by $\sinh(x)$ in the HR formula. This leads to the HRR formula.

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{kmax(n)} A_k(n)\ k^{1/2} \ \left[\frac{d}{dx}\frac{\sinh \frac{\pi}{k}(\frac{2}{3}(x-\frac{1}{24}))^{1/2}}{(x-\frac{1}{24})^{1/2}}\right]_{x=n}$$

- In the above, the sum over k is truncated in an n-dependent fashion at $kmax(n)$ – yet another HR innovation.

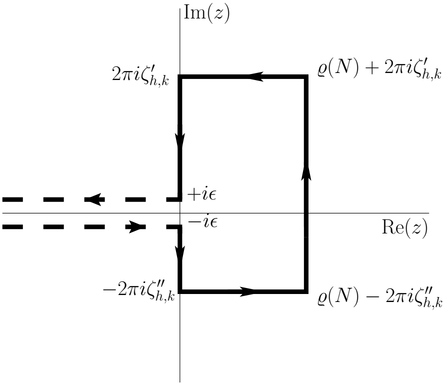
The Hardy-Ramanujan-Rademacher Formula for $p(n)$

- The major issue with the HR formula is that the sum over k is not convergent and the series is asymptotic.
- Rademacher was preparing to lecture on the HR formula for $p(n)$ in his course and started working through the proof. He discovered that a simple modification of the HR formula leads to a convergent sum! This proved the existence of an exact formula for $p(n)$ that Ramanujan always expected.
- In simple words, the change is to replace $\frac{1}{2}\exp(x)$ by $\sinh(x)$ in the HR formula. This leads to the HRR formula.

$$p(n) = (24n-1)^{-3/4} \sum_{k=1}^{kmax(n)} \frac{2\pi A_k(n)}{k}\ I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right) .$$

- Rademacher showed that integral in $\psi_{h,k}(n)$ could be converted to the representation of the modified Bessel function as an integral over the Hankel contour by adding exponentially suppressed terms that vanish as $N \to \infty$.

The contour for the modified Bessel function



$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \; \frac{1}{2\pi i} \int_\gamma \frac{1}{t^{\nu+1}} \exp\left[\frac{z^2 t}{4} + t^{-1}\right] \, \mathrm{d}t$$

γ is the Hankel contour shown in the above figure.

Applying the circle method for plane partitions

Recapping the steps

- The circle method provides the following expression for $pl(n)$:

$$p_2(n) = \lim_{N\to\infty} \sum_{(h,k)\in\mathcal{F}_N} \int_{z'_{h,k}}^{z''_{h,k}} \frac{\mathrm{d}z}{-2\pi i}\ e^{nz-2\pi i n\frac{h}{k}}\, \widehat{P}_2\left(e^{2\pi i\frac{h}{k}} e^{-z}\right)\ ,$$

$$=: \lim_{N\to\infty} \sum_{(h,k)\in\mathcal{F}_N} \psi_{h,k}(n) =: \sum_{k=1}^{\infty} \phi_k(n)\ .$$

- We thus need to study $P_2\left(e^{2\pi i\frac{h}{k}} e^{-z}\right)$ as $z\to 0$ and $\mathrm{Re}(z)>0)$. The generating function of the plane partition function is not a modular form and other methods need to be used.
- We need to then evaluate the integral.
- We need to study the convergence of the sum over k.

Theorem (Almkvist)

Let $z \to 0$ with $Re(z) > 0$ and $k \geq 1$, $1 \leq h < k$ and $(h,k) = 1$. Let $a = \zeta(3)$, $\omega_{h,k} = e^{2\pi i h/k}$, $B_p(x)$ be the periodic Bernoulli function and B_p be the p-th Bernoulli number. Then, for any integer $M \geq 1$ and $0 < \epsilon < 1$, we have

$$\log P_2\left(e^{-z}\omega_{h,k}\right) \sim \frac{a}{k^3 z^2} + \frac{k}{12}\log(zk) + k\zeta'(-1) + C_{h,k} + \sum_{p=1}^{M} v_{h,k}^{(p)}\, z^p + R_{h,k}^{(M)}(z)\ ,$$

where $C_{h,k}$ and $v_{h,k}^{(p)}$ $(p = 1, 2, \ldots)$ are generalised Dedekind sums.

$$C_{h,k} := \frac{k}{2}\sum_{j=1}^{k-1} B_2(j/k)\,\log\left|2\sin(\pi jh/k)\right|\ ,$$

$$v_{h,k}^{(p)} := \frac{(-1)^p k^{1+p}}{p!p(p+2)}\left[B_{p+2}B_p + \frac{p}{(2i)^p}\sum_{d=1}^{k-1} B_{p+2}(d/k)\cot^{(p-1)}(\pi dh/k)\right]\ ,$$

$$R_{h,k}^{(M)}(z) := \int_{-M-\epsilon-i\infty}^{-M-\epsilon+i\infty}\frac{ds}{2\pi i}\,(zk^2)^{-s}\,\Gamma(s)\sum_{d,d'=1}^{k}\omega_{h,k}^{dd'}\,\zeta(s-1,\tfrac{d'}{k})\,\zeta(s+1,\tfrac{d}{k})\ .$$

Remarks

- For the partition function, one saw that no terms of the form z^p (for $p > 1$) appeared in the Laurent series ($\zeta(2) = \frac{\pi^2}{6}$)

$$\log P(\omega_{h,k} e^{-z}) = \frac{\zeta(2)}{z k^2} + \frac{1}{2} \log(\tfrac{zk}{2\pi}) + is(h,k) - \frac{z}{24} + O(e^{-4\pi^2/zk})$$

- For the plane partition function, we see an infinite series with all powers of z appearing and generalised Dedekind sums appearing with each positive power of z.
- Let $V_{h,k}(z)$ denote the sum

$$V_{h,k}(z) := \sum_{p=1}^{M} v_{h,k}^{(p)}\, z^p + R_{h,k}^{(M)}(z) \ ,$$

  for some value of M that will be unspecified for the moment. Then, $R_{h,k}^{(M)}(z)$ is the remainder if we replace $V_{h,k}(z)$ by a series with M terms, thus explaining the notation.

The circle method for $p_2(n)$

Using the expansion given by Almkvist in the circle method leads to the following formal expression for $pl(n)$.

$$p_2(n) = \lim_{N\to\infty} \sum_{k=1}^{N} \sum_{\substack{h=1 \\ (h,k)=1}}^{k-1} \psi_{h,k}(n) \ , \quad \text{with}$$

$$\psi_{h,k}(n) = e^{-2\pi i n \frac{h}{k}} \cdot Z_{h,k}\, e^{V_{h,k}(D)} \cdot \widetilde{\mathcal{A}}_{\frac{k}{12}}\left(n\sqrt{ak^{-3}}\right) \ ,$$

where $Z_{h,k} = k^{-1}(a/k)^{\frac{1}{2}+\frac{k}{24}}\ e^{k\zeta'(-1)+C_{h,k}}$, $D = \frac{\mathrm{d}}{\mathrm{d}n}$ and the $\widetilde{\mathcal{A}}_\gamma(x)$ is given by integral

$$\widetilde{\mathcal{A}}_\gamma(x) = \int_{u''_{h,k}}^{u'_{h,k}} \frac{\mathrm{d}u}{2\pi i}\ u^\gamma \exp\left(\frac{1}{u^2} + xu\right) \ .$$

Here, $u'_{h,k} = z'_{h,k}\sqrt{ak^{-3}}$ and $u''_{h,k} = z''_{h,k}\sqrt{ak^{-3}}$.
The expression for $\psi_{h,k}(n)$ almost mirrors what we saw for the partition function.

A complication

- The main difference is the presence of a 'differential operator' $e^{V_{h,k}(D)}$ acting on the integral $\widetilde{\mathcal{A}}_{\frac{k}{12}}\left(n\sqrt{ak^{-3}}\right)$. The innovation of using this differential operator is due to Almkvist.
- In order to make the above formula for $p_2(n)$ useful, it is natural to push the contour of integration in the definition of $R_{h,k}^{(M)}(z)$ to the far left of the s-plane, i.e. take $M \to \infty$.
- This is the same as replacing $V_{h,k}(z)$ by the infinite sum $\widetilde{V}_{h,k}(z) := \sum_{p=1}^{\infty} v_{h,k}^{(p)}\, z^p$.
- Define $b_{h,k}^{(m)}$ as follows:

$$e^{\widetilde{V}_{h,k}(z)} =: \sum_{m=0}^{\infty} b_{h,k}^{(m)}\, z^m \ , \ \text{with} \ \ b_{h,k}^{(0)} = 1 \ .$$

- By analysing the behaviour of $b_{h,k}^{(m)}$ at large m, we show that the above series over m is an asymptotic one.
- Writing $\widetilde{\psi}_{h,k}(n)$ for the series obtained by replacing $V_{h,k}(z)$ by $\widetilde{V}_{h,k}(z)$ in $\psi_{h,k}(n)$, we show that $\psi_{h,k}(n) \sim \widetilde{\psi}_{h,k}(n)$.

The saddle-point estimate for the integral

- A saddle-point estimate for $\widetilde{\mathcal{A}}_\gamma(x)$ leads to the following expression for $x > 0$ and $\gamma > 0$ and let $\lambda := \dfrac{\gamma x^{-2/3}}{3\ 2^{1/3}}$.

$$\widetilde{\mathcal{A}}_\gamma(x) \sim \frac{1}{\sqrt{12\pi}}\,(x/2)^{-\gamma/3-2/3}\ e^{\left[3(\frac{x}{2})^{2/3}(1+f_1(\lambda))\right]} \times \Big[1 + f_2(\lambda)\Big]\ ,$$

  where $f_1(\lambda) = -\lambda^2 + \frac{\lambda^3}{3} + O(\lambda^5)$ and
  $f_2(\lambda) = -\frac{3\lambda}{2} + \frac{11\lambda^2}{8} + O(\lambda^3)$ as $\lambda \to 0^+$.
- Using this saddle-point estimate, the optimal truncation of the asymptotic series $\widetilde{\psi}_{h,k}(n)$ is determined to occur at $m := M^*(n,k)$ – this is called the superasymptotic truncation [Berry,1990].

$$\boxed{M^*(n,k) = \frac{c(\lambda)}{k}\, n^{1/3} - \frac{c(\lambda)^2}{4c_2 k}\, f_1''(\lambda)\ ,}$$

  where $c(\lambda) = 4\pi^2(2a)^{-1/3} e^{-\frac{1}{2}f_1'(\lambda)}$ and $c_2 = 3(a/4)^{1/3}$.

The superasymptotic formula for $\psi_{h,k}(n)$

Theorem (SG-NSP)

Let $c_1 = 2^{-1/4}(2a)^{1/36}\, e^{\zeta'(-1)}$, $c_2 = 3(a/4)^{1/3}$, $\lambda = \dfrac{k^2 n^{-2/3}}{24 c_2}$ *and* $c(\lambda) = 4\pi^2(2a)^{-1/3} e^{-\frac{1}{2} f_1'(\lambda)}$. *One then has the following superasymptotic formula:*

$$\psi_{h,k}(n) \sim e^{-2\pi i n h/k}\, Z_{h,k} \sum_{m=0}^{[M^*(n,k)]} b_{h,k}^{(m)}\, D^m \widetilde{\mathcal{A}}_{\frac{k}{12}}\left(n\sqrt{ak^{-3}}\right) + O\left(\mathcal{E}^*(n,k)\right) \;,$$

where

$$\mathcal{E}^*(n,k) = \frac{(2a)^{1/6}\, c_1^k\, (k^2 n^{-2/3})^{1+\frac{k}{24}}}{\pi^2 \sqrt{3k^3 M^*(n,k)}}\ \exp\left[\frac{1}{k}\left(-\frac{c(\lambda)^2}{4c_2} - c(\lambda)\, n^{1/3} + c_2\, n^{2/3}\right)\right] .$$

Remark: So the formula for $\psi_{h,k}(n)$ is not exact.

The Almkvist function

- Analogous to the HR formula, Almkvist guessed a function that he called $g_\gamma(x)$ that would replace the integral $\widetilde{\mathcal{A}}_\gamma(x)$. However, its divergent behaviour as $x \to 0$ implies that the sum over k will not be convergent even after carrying out the superasymptotic truncation.
- We show that a different function, that we call the Almkvist function, $\mathcal{A}_\gamma(x)$ can replace $\widetilde{\mathcal{A}}_\gamma(x)$ in a manner identical to Rademacher's modification of the HR formula.

Lemma (SG-NSP)

The Almkvist function $\mathcal{A}_\gamma(x) = \widetilde{\mathcal{A}}_\gamma(x) +$ *exp. suppressed terms.*

Both our function and the one chosen by Almkvist are solutions to the following third-order ordinary differential equation (a prime denotes $\mathrm{d}/\mathrm{d}x$ and γ is a real parameter)

$$x\ y'''(x) + (\gamma + 3)\ y''(x) - 2y(x) = 0\ .$$

The Almkvist function

- Analogous to the HR formula, Almkvist guessed a function that he called $g_\gamma(x)$ that would replace the integral $\widetilde{\mathcal{A}}_\gamma(x)$. However, its divergent behaviour as $x \to 0$ implies that the sum over k will not be convergent even after carrying out the superasymptotic truncation.
- We show that a different function, that we call the Almkvist function, $\mathcal{A}_\gamma(x)$ can replace $\widetilde{\mathcal{A}}_\gamma(x)$ in a manner identical to Rademacher's modification of the HR formula.

Lemma (SG-NSP)

The Almkvist function $\mathcal{A}_\gamma(x) = \widetilde{\mathcal{A}}_\gamma(x) +$ *exp. suppressed terms.*

The Almkvist fn. as a power series.

$$\mathcal{A}_\gamma(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{x^k}{k!\,\Gamma\left(\frac{3+\gamma+k}{2}\right)} \ ,$$

The Almkvist function

- Analogous to the HR formula, Almkvist guessed a function that he called $g_\gamma(x)$ that would replace the integral $\widetilde{\mathcal{A}}_\gamma(x)$. However, its divergent behaviour as $x \to 0$ implies that the sum over k will not be convergent even after carrying out the superasymptotic truncation.
- We show that a different function, that we call the Almkvist function, $\mathcal{A}_\gamma(x)$ can replace $\widetilde{\mathcal{A}}_\gamma(x)$ in a manner identical to Rademacher's modification of the HR formula.

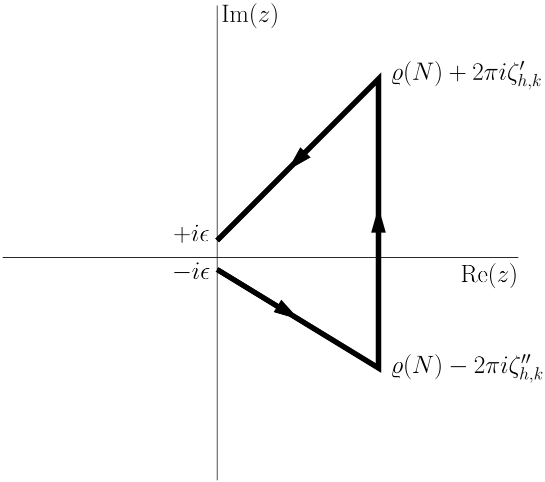
Lemma (SG-NSP)

The Almkvist function $\mathcal{A}_\gamma(x) = \widetilde{\mathcal{A}}_\gamma(x) +$ *exp. suppressed terms.*

The Almkvist fn. in terms of gen. hypergeometric functions.

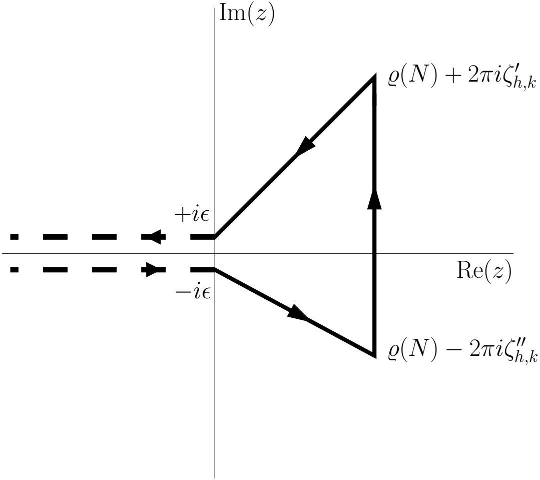
$$\mathcal{A}_\gamma(x) = \frac{1}{2}\left(\frac{{}_0F_2\left(;\frac{1}{2},\frac{3}{2}+\frac{\gamma}{2};\frac{x^2}{4}\right)}{\Gamma\left(\frac{3+\gamma}{2}\right)} + \frac{x\;{}_0F_2\left(;\frac{3}{2},2+\frac{\gamma}{2};\frac{x^2}{4}\right)}{\Gamma\left(\frac{4+\gamma}{2}\right)}\right)\,.$$

The contour for the Almkvist function



This is the contour that appears in the integral representation of $\mathcal{A}_\gamma(x)$

The contour for the Almkvist function



The Hankel contour appears in the integral representation of $g_\gamma(x)$, the function used by Almkvist.

The superasymptotic formula for $p_2(n)$

Theorem (SG-NSP)

Let $\psi_{h,k}(n)$ be as in the previous Theorem with $\widetilde{\mathcal{A}}_\gamma(x)$ replaced by $\mathcal{A}_\gamma(x)$. The superasymptotic formula for the number of plane partitions of n is

$$p_2(n) \sim \sum_{k=1}^{[kmax(n)]} \sum_{\substack{h=1 \\ (h,k)=1}}^{k-1} \psi_{h,k}(n) + \mathcal{O}(n^{-1/6}) \ ,$$

where $kmax(n) = 2.948\ n^{1/3} - 0.979 \log n + 1.587$.

- Since the sum over k is convergent, we can cut-off the sum at $k = kmax(n)$. This parametric cut-off is originally due to Hardy and Ramanujan who cut-off the sum over k for $p(n)$ in the HR formula at $kmax(n) = cn^{1/2}$ for some constant c.
- We determine that $kmax(n)$ is the cut-off when the truncation error is $O(n^{-1/3})$.

Example

k	$\phi_k(750)$
1	2545743024358645039521920749024859571789657217789975418420497702709720.3001
2	1169353378721087578836884133296412.0536
3	1308038187203153215044.2870
4	−766248063769796.4865
5	249747729385.7157
6	258376791.8757
7	−3577528.9989
8	−1684.4658
9	−13708.6585
10	1766.7342
11	−274.7588
12	−61.8573
13	−6.9383
14	0.409 4
15	2.5409
16	−0.1379
17	−0.4472
18	−0.1105
19	−0.0038
Total	2545743024358645039521920749024859572959010596512371034678586927966061.0530
Exact	2545743024358645039521920749024859572959010596512371034678586927966061.0000

Table: Numerical evaluation of $p_2(750)$.

Remark: Our formula is exact for all $n \leq 6400$ as the superasymptotic truncation error is < 1.

Concluding Remarks

- Higher dimensional partitions are interesting combinatorial objects and as their dimension increases, the lesser we know about them.
- The appearance of oscillations in the asymptotics of solid partitions provides a new insight that is worth understanding.
- The algorithm of Bratley-McKay can be easily modified to enumerate entries in the A-matrix. However, we lack an algorithm that directly enumerates entries in the F-matrix. There appears to be interesting connections with graph theory!
- There is a well-developed theory of hyperasymptotics that improves on the superasymptotic truncation that we discussed. We have been able to compute $p_2(10000)$ using this method.

Hopefully, I've got a few of you interested in this fascinating topic of higher dimensional partitions.

Thank you

1. A. O. L. Atkin, P. Bratley, I. G. Macdonald, and J. K. S. McKay, "Some computations for m-dimensional partitions," *Proc. Cambridge Philos. Soc.* **63** (1967) 1097–1100.
2. D. P. Bhatia, M. A. Prasad, and D. Arora, "Asymptotic results for the number of multidimensional partitions of an integer and directed compact lattice animals," J. Phys. A **30** no. 7, (1997) 2281–2285.
3. G. Almkvist, *Exact Asymptotic formulas for the coefficients of nonmodular Functions*, J. of Numb. Th. **38** (1991) 145–160.
4. G. Almkvist, *A rather exact formula for the number of plane partitions*, Cont. Math. **145** (1993), 21–26.
5. G. Almkvist, *Asymptotic Formulas and Generalized Dedekind Sums*, J. Exp. Math. **7** (1998), 343–359.
6. G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. of the LMS **2** (1918), 75–115.
7. H. Rademacher, *A convergent series for the partition function* $p(n)$, Proc. of the National Academy of Sciences **23** (1937), no. 2, 78–84.